Fermion Zero Modes for Abelian BPS Monopoles

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Abstract

Fermion zero modes for abelian BPS monopoles are considered. In the spherically symmetric case the normalisable zero modes are determined for arbitrary monopole charge N. If $N > 1$ the zero modes are zero along $N-1$ half-lines emanating from the monopole.

Keywords: abelian gauge theory, BPS monopoles, Weyl equation, fermion zero modes.

Fermion zero modes for BPS monopoles can be constructed via the same Nahm transform used to obtain the monopoles [1]. The construction is, however, cumbersome for magnetic charge $N > 2$. In this letter we obtain zero modes for abelian BPS monopoles. Our approach is to directly integrate the Weyl equations in three-dimensional space rather than use Nahm's method (which has been adapted to abelian monopoles in [2]).

The abelian BPS equations read

$$\mathbf{B} = \nabla\Phi, \tag{1}$$

where Φ is a real Higgs field and $\mathbf{B}$ is a magnetic field derived from a vector potential $\mathbf{A}$. The Maxwell equation $\nabla \cdot \mathbf{B} = 0$ implies that the Higgs field Φ

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obeys the Laplace equation. The Higgs field

$$\Phi = \frac{g}{2\pi}\left(a - \frac{1}{2}\sum_{i=1}^{N}\frac{1}{|\mathbf{r}-\mathbf{r}_i|}\right), \tag{2}$$

with a and g constant, is harmonic away from N singularities $\mathbf{r}_i$ $(i = 1, 2, ..., N)$. Physically, the system comprises N Dirac monopoles each with magnetic charge g interacting with a Higgs field. Here a fixes the asymptotic value of the Higgs field. Consider the Weyl operators

$$D = eI_2\Phi + i\boldsymbol{\sigma}\cdot(-i\nabla + e\mathbf{A}) \qquad D^\dagger = eI_2\Phi - i\boldsymbol{\sigma}\cdot(-i\nabla + e\mathbf{A}) \tag{3}$$

where e is the electric charge of the fermion and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. These Weyl operators assume a real Yukawa coupling in Minkowski space. However, identifying Φ as A_0 they are also Weyl operators for self-dual monopoles defined in Euclidean space.

The Dirac quantisation condition requires $eg = 2\pi n$ with n integer. If

$$eg = 2\pi$$

and $a > 0$, $D^\dagger$ has N normalisable zero modes. In the $N = 1$ case we have

$$\Phi = \frac{g}{2\pi}\left(a - \frac{1}{2r}\right), \qquad \mathbf{A} = \frac{g}{4\pi}\frac{y\mathbf{e}_x - x\mathbf{e}_y}{r(r-z)} = -\frac{g(1+\cos\theta)\mathbf{e}_\phi}{4\pi r\sin\theta}, \tag{4}$$

taking the origin as the location of the monopole and (r, θ, ϕ) denote spherical polar coordinates. Here the Dirac string lies on the positive z-axis.

One can verify that

$$\psi = \frac{e^{-ar}}{\sqrt{r}}\begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2)e^{i\phi} \end{pmatrix} \tag{5}$$

is a zero mode of $D^\dagger$ (the components of $D^\dagger$ in spherical polar coordinates are given in the appendix). D has no zero modes. This result can be obtained by taking the large r limit of the Jackiw-Rebbi zero mode [3, 4] for the basic $SU(2)$ BPS monopole after performing a gauge transformation which diagonalises the Higgs field. The fermion density $\psi^\dagger\psi = e^{-2ar}/r$ is spherically symmetric and has an integrable singularity at the monopole centre.

The general N case is more complicated. However, in the spherically symmetric case where the positions of the N monopoles coincide the Higgs field is

$$\Phi = \frac{g}{2\pi}\left(a - \frac{N}{2r}\right) \tag{6}$$

and $\mathbf{A}$ is the $N = 1$ potential multiplied by N. Here $D^\dagger$ has N normalisable zero modes:

$$\psi^m = r^{\frac{1}{2}(N-2)} e^{-ar} \begin{pmatrix} -\sin^{N-m+1}(\theta/2)\cos^{m-1}(\theta/2)e^{i(m-1)\phi} \\ \sin^{N-m}(\theta/2)\cos^m(\theta/2)e^{im\phi} \end{pmatrix} \quad m = 1, 2, ..., N. \tag{7}$$

These solutions resemble (in particular with respect to the θ and ϕ dependence) known solutions of the Dirac equation in the background of abelian monopoles [5]. However, our solutions are written directly in terms of trigonometric functions rather than spherical harmonics[1]. Our solutions are normalisable with L^2 norm $4\pi(N-m)!(m-1)!(2a)^{-(N+1)}$. As the zero modes ψ^m are annihilated by the Hamiltonian it is not clear to us whether the presence of the Higgs field cures the self-adjointness problem [5, 7] associated with monopole Hamiltonians. To address this issue one needs to study the scattering states [5].

Note that the densities $\psi^{m\dagger}\psi^m$ are not spherically symmetric for $N > 1$. For $N = 2$ the first zero mode is zero along the positive z-axis while the second mode is zero on the negative z-axis. By taking a suitable linear combination of ψ^1 and ψ^2 one can obtain a zero mode with a zero along any half-line emanating from the monopole; the zero mode is axially symmetric about the axis on which the half-line lies. The $N > 1$ zero modes are zero along $N-1$ half-lines. Our ψ^1 and ψ^N have zeros of strength $N-1$ on the positive and negative z-axes, respectively; the remaining $N-2$ modes have lower strength zeros on both the positive and negative z axes. Again one can adjust the directions of the $N-1$ half lines by taking different linear combinations of the N zero modes. For a discussion of zeros of fermion zero modes in a different context see [8].

In general, the $N > 2$ zero modes are not axially symmetric even though the ψ^m are all axially symmetric about the z-axis. If Ψ is a linear combination of the ψ^m, the density $\Psi^\dagger\Psi$ has the form

$$\Psi^\dagger\Psi = r^{N-2}e^{-2ar}f(\theta, \phi), \tag{8}$$

where f is a function of θ and ϕ. For $N > 2$ one can see that the zero modes vanish at the position of the monopole and the zero modes peak somewhere on the sphere $r = (N-2)/2a$. The function $f(\theta, \phi)$ has up to $N-1$ zeros; if there are fewer than $N-1$ zeros these are repeated zeros associated with coincident half-lines. Examination of $f(\theta, \phi)$ for several zero modes indicates that $f(\theta, \phi)$ has a single peak. For example, the $N = 3$

[1] If N is even the zero modes can be expressed in terms of standard spherical harmonics. If N is odd spin-weighted or monopole harmonics [6] are required.

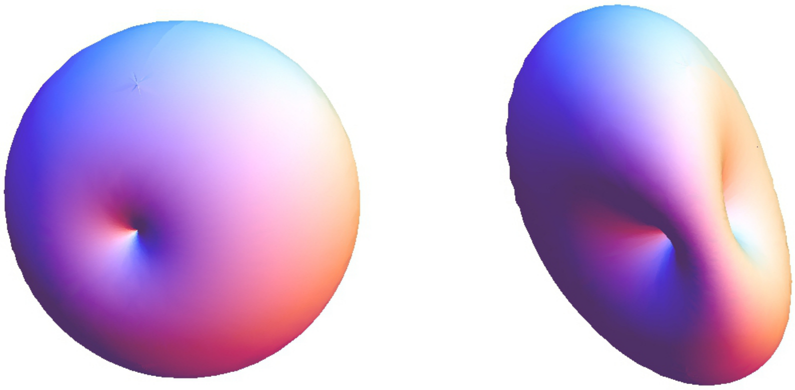

Figure 1: Spherical plots of $f(\theta,\phi)$ for an $N=2$ and $N=3$ zero mode (the scale depends on the normalisation and hence a). The left plot shows $f(\theta,\phi)$ for an $N=2$ zero mode ($\Psi=\psi^1+\psi^2$) which has one zero and is axially symmetric. Two zeros are visible for the $N=3$ zero mode ($\Psi=\psi^1+\psi^2-i\psi^3$) on the right. Both zero modes have a single peak.

zero mode $\Psi=\psi^1+\psi^2+\psi^3$ yields an $f(\theta,\phi)$ function with two zeros and a maximum on the equator $\theta=\pi/2$ (these three points are equally spaced on the equator). The maximum of $f(\theta,\phi)$ is a point on the unit sphere except for some axially symmetric solutions where $f(\theta,\phi)$ peaks on a circle. In Figure 1 spherical plots of $f(\theta,\phi)$ are given for an $N=2$ and an $N=3$ zero mode.

van Baal [9] has given an ansatz which provides a solution of the Weyl equation for any solution of the abelian BPS equation:

$$\psi_{vB}=D\begin{pmatrix} w \\ 0 \end{pmatrix} \tag{9}$$

satisfies $D^\dagger\psi_{vB}=0$ where

$$\Phi=\frac{g}{2\pi}\frac{\partial}{\partial z}\log w,\qquad \mathbf{A}=-\frac{g}{2\pi}\mathbf{k}\times\nabla\log w \tag{10}$$

and $\log w$ satisfies the Laplace equation. This works as the BPS equation implies $D^\dagger D$ is a scalar and (10) gives $D^\dagger D w=0$. Taking $\log w=\frac{1}{2}N\log(r-z)+az$ yields our Φ and $\mathbf{A}$. However, ψ_{vB} is not normalisable[2] though for $a=0$ it agrees with our zero mode ψ^1. Indeed, ψ^1 is the

[2] Remarkably, (9) does provide one normalisable zero mode for a different class of Higgs fields; here Φ has $2N$ singularities representing N positively charged and N negatively charged monopoles [9, 10].

$a = 0$ van Baal solution multiplied by e^{-ar}. As the van Baal construction does not rely on spherical symmetry this approach may provide information about the $a = 0$ limit of the general case where the N monopoles are separated.

We have considered fermion zero modes for BPS monopoles and have obtained solutions for arbitrary magnetic charge N. The spherically symmetric abelian case we have solved may provide a model for non-abelian magnetic bags; although higher charge $SU(2)$ BPS monopoles are never spherically symmetric, solutions with approximate spherical symmetry may emerge for large N [11]. It would be interesting to investigate the extent to which our zero modes approximate the fermion zero modes of magnetic bags.

Appendix

The components of $D^\dagger$ associated with (6) are ($eg = 2\pi$)

$$
\begin{aligned}
(D^\dagger)_{11} &= a - \frac{N}{2r} - \cos\theta\,\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \\
(D^\dagger)_{12} &= e^{-i\phi}\left[-\sin\theta\,\frac{\partial}{\partial r} - \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{i}{r\sin\theta}\frac{\partial}{\partial\phi} + \frac{N}{2}\frac{(1+\cos\theta)}{r\sin\theta}\right] \\
(D^\dagger)_{21} &= e^{i\phi}\left[-\sin\theta\,\frac{\partial}{\partial r} - \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{i}{r\sin\theta}\frac{\partial}{\partial\phi} - \frac{N}{2}\frac{(1+\cos\theta)}{r\sin\theta}\right] \\
(D^\dagger)_{22} &= a - \frac{N}{2r} + \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}.
\end{aligned}
$$

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